Non-Homogeneous Wavelet Denoising

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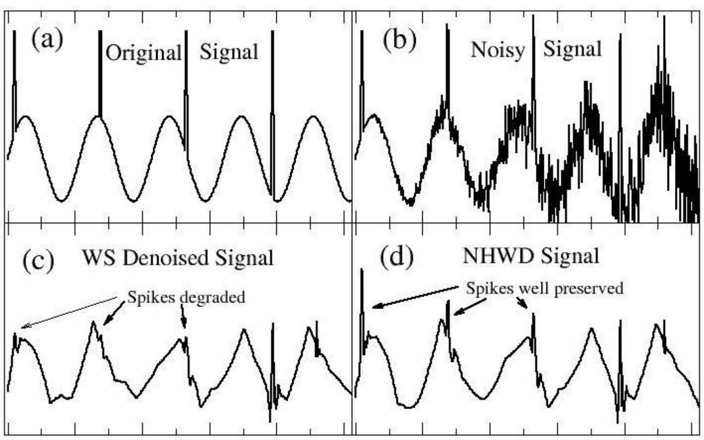
Introduction

Wavelet denoising has been widely used in signal processing to remove noise while preserving edges and spikes in the data, with minimal smoothing. A standard approach is wavelet shrinkage (WS) of the wavelet coefficients, d_j, $j = 1,\ldots,n$, [1,2] using a threshold $t = \sigma\sqrt{2 \log n}$, a decision theoretic result based on risk minimization. Another approach is to treat thresholding as a hypothesis testing problem [3,4], with null hypothesis H_0: $d_j^t = 0$. Both approaches address the case of signals corrupted by additive and spatially homogeneous noise. Often, noise properties are spatially dependent, e.g., in diffusion tensor MRI (DTI), the components of the measured diffusion tensor as well as quantities derived from it, exhibit spatially variable noise properties [5]. Here we introduce Non-Homogeneous Wavelet Denoising (NHWD), which assumes that the variance field, $\sigma^2(x,y,z)$, is known. Even though there are alternative and simpler ways for denoising diffusion tensor images [6], NHWD also offers a direct way to approximate DT data using wavelet basis. More importantly, we envision that NHWD will be used in other signal processing and imaging applications, for which the assumption of homogeneous noise is not satisfied.

Theory

The goal of NHWD is to estimate the true signal, y_i^t, given an observed signal y_i, $i = 1,\ldots,n$, corrupted by additive and spatially inhomogeneous noise, with variance, σ_i according the equation, $y_i = y_i^t + \sigma_i \varepsilon_i$. We assume ε_i are normally distributed, $N(0,1)$, but this condition can be extended to a general case of random variables with zero mean and unit variance. NHWD treats thresholding the wavelet coefficients as multiple tests of the null hypothesis, H_0: $d_j^t = 0$ against the alternative H_1: $d_j^t \neq 0$. The d_js are related to the observed data through a linear transformation: $d_j = \sum_i w_{ij} y_i$. Since w_{ij} are the elements of an orthonormal wavelet transformation matrix, $W=[w_{ij}]$, the d_js can be written in terms of the true, or denoised, wavelet coefficients, d_j^t, as: $d_j = d_j^t + \sigma_j \varepsilon_j$, where ε_j are also $N(0,1)$. The σ_j are related to the original σ_i through: $\sigma_j^2 = \sum_i w_{ij}^2 \sigma_i^2$ (1)

The NHWD procedure consists of five steps: **(1)** Determine wavelet coefficients, d_js; (**2**) find σ_js from the original σ_i using Eq. 1, and for each d_j find the p-value in testing H_0, $p_j = 2(1 - \Phi(|d_j|/\sigma_j))$ where Φ is the cumulative normal distribution; **(3)** Control the dissipation of statistical power of the multiple tests by rank ordering the p_j by increasing magnitude, $p_{(1)} \leq p_{(2)} \leq \ldots \leq p_{(n)}$, and find the highest rank k for which $p_{(k)} < k\alpha/n$ where α is the confidence level [3,4,7]; **(4)** Threshold the j^{th} wavelet coefficient, d_j, using threshold t_j, such that $t_j = \sigma_j \Phi^{-1}(1 - p_{(k)}/2)$; **(5)** Use the thresholded wavelet coefficients d_i^t , and the inverse wavelet transform to obtain the denoised data, y_i^t.



Methods and Results

We first test our scheme on a simulated signal consisting of a pure sine (perfect frequency localization) together with several narrow spikes (localized in time). To this “oracular” signal (Figure 1a) normal random noise is added, (Figure 1b), such that the standard deviation at a particular time point is equal to $\sigma(t) = \sigma_0 t^{6/5}$. Figures 1c and 1d compare the results obtained using standard wavelet shrinkage (WS) due to Donoho and NHWD. The Donoho scheme is conservative, i.e., it removes noise only with high probability. However, for the high noise regions its performance is similar to NHWD with α=0.05, in the low noise region it introduces significant smoothing, i.e., the spikes are almost completely degraded. NHWD preserves edges/spikes in the low noise regions while yielding comparable denoising in smooth regions. We also applied this methodology to DTI data (see discussion).

Discussion

Generally, obtaining $\sigma_i(\mathbf{x})$ is the most challenging task in NHWD. In DTI one can obtain theoretically predicted values of $\sigma_i(\mathbf{x})$ for each component, but a more robust and pragmatic approach is to obtain bootstrap estimates of $\sigma_i(\mathbf{x})$ for every voxel in the image [5]. Besides applying NHWD to the components of the diffusion tensor $\boldsymbol{D}$, the “noise free” coefficients, d_j^t can be used to construct a continuous representation of $\boldsymbol{D}(\mathbf{x})$ using wavelets rather than B-splines, as the basis [8]. N.B. the standard multi-resolution analysis applies the wavelet filters only to the coefficients of the expansion in terms of the scaling functions. To use NHWD for constructing a continuous approximation of $\boldsymbol{D}(\mathbf{x})$, one needs either to modify the wavelet basis or the linear transformation matrix (W) to also include this additional linear transform (e.g. direct B-spline transform).

Conclusion

We tested a new methodology for denoising data with inhomogeneous noise properties and showed that it successfully treats cases in which the variance of the noise is spatially dependent. An extension to this work can be made for cases of an arbitrary probability distribution of noise, as well as the mixture of different noise sources.

[1] DL Donoho, IM Johnstone, *Biometrika*, **81**(3):425, 1994.
[2] DL Donoho, *IEEE Trans Info Theory*, **41**(3):613, 1995.
[3] F Abramovich and Y. Benjamini, *Statist. Data Anal.*, 22, pp. 351-361, 1996.
[4] B. Vidakovic, *Statistical Modelling by Wavelets*; Wiley & Sons, 1999.
[5] S.Pajevic, P.J.Basser, *J Magn. Reson.* **161**, pp. 1-14 , 2003.
[6] U Nevo, Bat Sheva Workshop on Diffusion, Tel Aviv, Israel, 2001.
[7] Y Benjamini, Y. Hochberg, *J. Roy. Statist. Soc. Ser. B*, 57, 289-300, 1995.
[8] S Pajevic, A. Aldroubi, P.J. Basser, *J Magn Reson* **2002**, *154*, 85.