Size and Shape Matter: Another Look at Tensor Statistics

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Introduction

Understanding biological processes involved in the growth, development and aging of brain circuitry is of central importance in neurobiology, neuroanatomy and neuropathology. New insights can be gleaned through further study of the statistical properties of the diffusion tensor and the diffusion tensor field. A recently proposed joint probability distribution of the eigenvalues and eigenvectors of the diffusion tensor[1,2] provides a framework for characterizing the variability of the diffusion tensor and tensor-derived quantities within an imaging volume or a Region of Interest (ROI). This distribution is parametrized by a local 2nd-order mean diffusion tensor and a 4th-order covariance tensor. The latter has not been studied in the context of segmenting, clustering or classifying tissues, nor have its various symmetry properties, spatio-temporal, intra- or inter-subject dimensions been explored or exploited for these purposes.

Theory and Methods

One contribution of this work is to recast the covariance tensor so that its components relate to fluctuations in random variables characterizing the size and shape of the diffusion ellipsoid, its prolate/oblate forms and higher-order size and shape combinations. This covariance tensor can also exhibit many types of internal symmetry similar to those found in crystals and other structures whose material properties are also described by 4th-order tensors of the same form[3-7]. Further analysis of these symmetry properties has the potential to reveal new structural features of tissue. Statistical techniques can be applied to determine which symmetries and symmetry classes are supported by particular, experimentally measured covariance tensors associated with DTI data collected at the same or different locations within a brain ROI (spatial) for one or more individuals at the same or different measurement times (temporal) or both (spatio-temporal), and even across platforms and across sites. This novel avenue has the potential for providing new intrinsic features or "stains" for characterizing similarities and differences between tissues.

	ADC′	D'_{xx}	D'_{yy}	D'_{zz}	$\sqrt{2}D'_{xy}$	$\sqrt{2}D'_{xz}$	$\sqrt{2}D'_{yz}$
	Size Only				Size and Shape		
ADC′	1	1	1	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
D'_{xx}	1	1	1	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
D'_{yy}	1	1	1	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
D'_{zz}	1	1	1	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
	Size and Shape				Shape Only		
$\sqrt{2}D'_{xy}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	2	2	2
$\sqrt{2}D'_{xz}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	2	2	2
$\sqrt{2}D'_{yz}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$	2	2	2

Figure 1. The general partition of size-shape components of the local covariance tensor. The cross-products of tensor elements in the table are implicit and deducible by noting the row and column in which each resides.

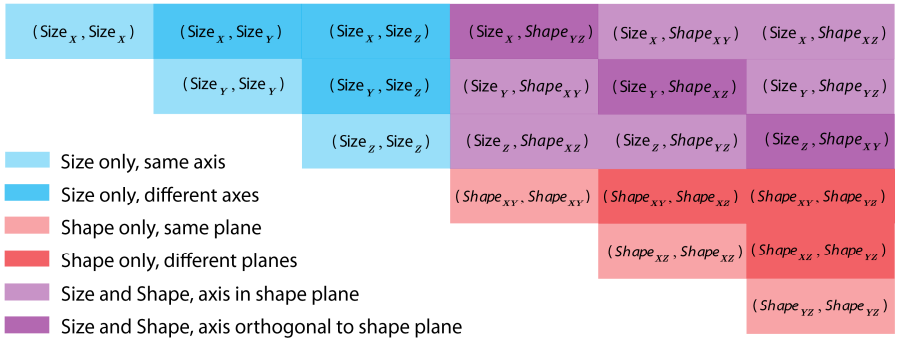

Figure 2. Size, shape and size-shape effects of forces external and internal to the second-order diffusion tensor in its inherent coordinate system (adapted from Helbig, 1994.)

Results

Figure 1 shows the blocks of the matrix form of the 2nd-order covariance tensor (matrix)[8] associated with the 4th-order covariance tensor described above. This 2nd order tensor arises when one writes the mean-centered tensor elements in vector form as $\underline{D'} = (ADC', D'_{xx}, D'_{yy}, D'_{zz}, \sqrt{2}D'_{xy}, \sqrt{2}D'_{xz}, \sqrt{2}D'_{yz})$, where D′ is the deviation or deviatoric tensor. This particular choice of a vector random variable allows one to more easily visualize and identify different blocks of this matrix as being associated with the diffusion tensor's size, shape and size-shape. Figure 2 shows the four types of additional symmetry evident when each symmetric block is examined more closely. These higher-order size and shape effects are classifiable according to which axes and planes of the tensor's inherent three-dimensional coordinate system are affected. The number of distinct elements (free parameters) in this matrix can vary from two (corresponding to a fully isotropic medium) to 21 (corresponding to complete anisotropy). Each additional free parameter starting from three defines a different tensor symmetry class. In contrast to other fields of application, the symmetry classes of diffusion tensors and diffusion tensor fields are not known *a priori*. In crystallography, for instance, the symmetry class of a crystal is often known prior to the determination of its physical properties[6]. Mechanical testing can often help determine the symmetry of a material from its stress-strain behavior. It is not currently known what symmetry classes one will measure from DTI in different tissues and tissue types from an examination of the local covariance tensor.

Discussion and Conclusion

More biological information may already be contained in diffusion tensors measured typically by standard protocols run on clinical scanners in addition to that conveyed by their first-order size (e.g., mean, axial and radial diffusivity), second-order shape (e.g., FA) and third-order shape (skewness) coefficients. An enriched understanding of the "size" and "shape" of a tensor can be obtained by close inspection of its associated covariance tensor. This tensor takes on a variety of patterns whose complexity depends on the tissue's unknown symmetry class, which can be determined by statistical methods. The experimental frame by which the tensor is measured *in situ* (how it is "seen") by a particular sampling design can have profound effects on which size and shape features can and cannot be seen. Realistic simulations using well understood signal-generation models and sampling schemes[9] and clinical research applications to brain-based disorders, such as autism, are needed to evaluate the practical utility of these ideas in biology and medicine.

Bibliography

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