Optimal Design of DT-MRI Experiments Using a New Tensor-variate Gaussian Distribution

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SYNOPSIS:

We propose that a necessary condition for an optimally designed DT-MRI experiment is that a new normal distribution for the estimated diffusion tensor, $p(\boldsymbol{D}) \propto \exp(-1/2\ \boldsymbol{D}:\boldsymbol{A}:\boldsymbol{D})$, be characterized by an isotropic 4th-order precision tensor, $\boldsymbol{A}$. This requirement ensures that statistical properties of $\boldsymbol{D}$ (and quantities derived from it) are all rotationally invariant. We propose a measure of anisotropy of $\boldsymbol{A}$, which can be used to assess the degree of rotational invariance of any DT-MRI experimental design. We apply this measure to evaluate various DT-MRI experimental designs.

THEORY AND RESULTS:

Batchelor recently proposed that a diffusion weighted acquisition scheme in which diffusion gradients were oriented toward the vertices of an icosahedron possessed rotationally invariant statistical properties by showing that the design matrix for this scheme is the same as one obtained when using an infinite number of gradient vectors that are uniformly distributed on a unit sphere (1).

Here we propose a general framework for understanding and assessing the degree of rotational invariance of any DT-MRI experimental design. First, we introduce a new *tensor*-variate normal distribution, $p(\boldsymbol{D})$, that describes the variability of the estimated 2nd-order diffusion tensor, $\boldsymbol{D}$, in an ideal DT-MRI experiment (2). In Eq. (1), $\boldsymbol{A}$ is a 4th-order precision tensor, and $\hat{D}$ is the mean diffusion tensor. When $\boldsymbol{A}$ is a general 4th-order isotropic tensor (see (3)), $\boldsymbol{A}^{iso}$, its form is given by Eq. (2) where μ and λ are constants, and $\boldsymbol{\delta}$ is the 2nd-order isotropic tensor. Second, we propose that isotropy of $\boldsymbol{A}$ implies that the statistical properties of $p(\boldsymbol{D})$ are independent of the choice of laboratory coordinate system in which $\boldsymbol{D}$ is measured. This condition ensures that the experimental design is rotationally invariant. In a rotationally invariant DT-MRI experiment, $\boldsymbol{D}$ are distributed according to Eq. (3), which is obtained by substituting Eq. (2) into Eq. (1). Third, to assess the degree to which a DT-MRI experimental design deviates from statistical isotropy we measure the "distance" between $\boldsymbol{A}$ for a particular design and $\boldsymbol{A}^{iso}$. One proposed measure of the degree of experimental design anisotropy (EDA) is given in Eq. (4). The EDA is normalized to one, is unitless, and is zero when $\boldsymbol{A} = \boldsymbol{A}^{iso}$. The distance norm, ‖…‖ in Eq. (4) could be Euclidean or Cityblock, summing over all 81 elements of $\boldsymbol{A}$. Fourth, we generate empirical estimates of $\boldsymbol{A}$ and EDA for any experimental design using Monte Carlo (MC) methods.

$$p(D) \propto \exp\left(-\frac{1}{2}\sum_{i,j,m,n=1}^{3}\left(D_{ij}-\hat{D}_{ij}\right)A_{ijmn}\left(D_{mn}-\hat{D}_{mn}\right)\right) \quad (1)$$

$$A^{iso}_{ikpm} = \lambda\delta_{ik}\delta_{mp} + \mu\left(\delta_{im}\delta_{kp} + \delta_{ip}\delta_{km}\right) \quad (2)$$

$$p(D) \propto \exp\left(-\frac{1}{2}\left\{\lambda\left(Trace\left(D-\hat{D}\right)\right)^2 + 2\mu\ Trace\left(\left(D-\hat{D}\right)^2\right)\right\}\right) \quad (3)$$

$$EDA(\lambda,\mu) = \left\| A - A^{\mathbf{iso}} \right\| / \left\| A \right\| \quad (4)$$

Results from MC simulations of $\boldsymbol{D}$ data obtained using icosahedral (4) and dodecahedral (5) gradient schemes are shown in the Table below. The assumed $\boldsymbol{D}$ is isotropic having the diffusivity of brain gray matter (< D > = 700 mm^2/sec). N.B. using an isotropic diffusion tensor does not imply that its statistical properties will be rotationally invariant. Although the design matrix for these schemes is predicted to be statistically isotropic (1), these MC simulations produce a non-zero EDA, however, it decreases as the number of acquisitions increases (see Table below). This drop in EDA with increasing number of acquisitions is much smaller in schemes whose design matrix is not isotropic. Increasing the number of non-diffusion weighted images in an experiment also does not significantly alter the estimated EDA in these designs.

	Icosahedral (6)					Dodecahedron (10)			
Number of Acquisitions	6	12	30	60		10	20	30	60
Anisotropy (EDA)	0.467	0.201	0.069	0.034		0.259	0.115	0.074	0.035

DISCUSSION:

An advantage of using this new tensor-variate probability distribution to design DT-MRI experiments is that one can consider gradient schemes having different numbers of gradient acquisitions, gradient strengths, and gradient magnitudes. Using this framework, we can show that any combination (concatenation) of rotationally invariant experimental designs will result in a rotationally invariant experimental design. This allows us to build rotationally invariant experiments out of existing rotationally invariant experimental "building blocks".

More generally, this new normal distribution for a tensor-valued random variable can be used to determine the effect of a rotation of the laboratory coordinate system or the application of a general affine transformation to $\boldsymbol{D}$ (as is required in some image warping and registration applications (6)) on the distribution of $\boldsymbol{D}$, and quantities derived from it. $p(\boldsymbol{D})$ can also provide estimates of the distribution of the apparent diffusion coefficient (ADC), p(ADC), obtained by projecting the $\boldsymbol{D}$ along a particular direction, and of p(Trace($\boldsymbol{D}$)). It can also be used to predict the distribution of the principal diffusivities (eigenvalues) and principal directions (eigenvectors) of $\boldsymbol{D}$, as well as scalar invariants of $\boldsymbol{D}$ (e.g., Trace($\boldsymbol{D}$)^2 and Trace($\boldsymbol{D}^2$)) that characterize different features of anisotropic diffusion. Finally, the constraint that $\boldsymbol{A}$ is an isotropic 4th-order tensor can be used in conjunction with other objective functions (performance measures) to obtain an "optimal" experimental design.

CONCLUSIONS:

The new normal distribution given in Eq. (1), which describes the variability of 2nd-order tensor data, should be useful in feature extraction, prediction, estimation, filtering, and hypothesis testing applications. Here we use this new distribution to assess the degree of rotational invariance or statistical isotropy of DT-MRI experiments.

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