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Analytical Expression for the Gradient b-Factor in NMR Diffusion Imaging.

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Introduction:

Measurements of molecular diffusion using MRI has generated great interest, especially in functional assessment, tissue characterization, and treatment monitoring[1]. The estimation of the diffusion tensor, $\underline{D}$, in anisotropic media by NMR spectroscopy and imaging requires the determination of a b-matrix from the gradient pulse sequence[2]:

$$\ln\left(\frac{S(\underline{b})}{S(0)}\right) = -\sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij} D_{ij} \qquad [1]$$

where b_{ij} are components of the b-matrix, $\underline{b}$; D_{ij} are components of the effective self-diffusion tensor, $\underline{D}$; and $S(\underline{b})$ and $S(0)$ are echo intensities with and without applied diffusion gradients, respectively.

We present analytical expressions of the b-matrix for a 2D-FT *spin-echo imaging sequence, and show the effect of ignoring* cross-terms in estimating the apparent diffusion tensor.

Expression for the b-factors:

Using definitions for the gradient pulse sequences[3,4] : $\mathbf{G}(t) = (G_X(t), G_Y(t), G_Z(t))$, $\mathbf{F}(t) = \int_t^{TE} \mathbf{G}(t')dt'$, and $f = \mathbf{F}(1/2\,TE)$, we relate the echo intensity to the diffusion tensor[3,4] :

$$\ln\left(\frac{S(\underline{b})}{S(0)}\right) = -\gamma^2 \int_0^{TE} \left(\mathbf{F}(t)-2\xi(t)\mathbf{f}\right) \underline{D} \left(\mathbf{F}(t)-2\xi(t)\mathbf{f}\right)^T dt, \qquad [2]$$

where TE is the echo time, γ is the gyromagnetic ratio, $\xi(t < 1/2\,TE) = 0$, $\xi(t \geq 1/2\,TE) = 1$. The b-matrix elements are derived from the expression[2]:

$$\underline{b} = \gamma^2 \int_0^{TE} \left(\mathbf{F}(t)-2\xi(t)\mathbf{f}\right)^T \left(\mathbf{F}(t)-2\xi(t)\mathbf{f}\right) dt. \qquad [3]$$

A symbolic manipulation program similar to that employed by Price et al.[5], was used to derive analytical expression for the b-matrix. A generalized 2D-FT spin-echo imaging sequence, shown in Figure 1, was synthesized from a library of sinusoidal and trapezoidal gradient pulses, which includes all gradient pulses that are typically encountered. The imaging gradient intensities are: Gsl, 90° slice-selection gradient; Gpe, Grdp, or Gsrf, phase-encoding, read-dephasing or slice-refocusing gradients, respectively; Gd_r, Gd_p or Gd_s, diffusion gradients in the r = read, p = phase, and s = slice directions, respectively; Gc_r, Gc_p, or Gc_s, crusher gradients in the read, phase, and slice directions, respectively; 1/2 Gsl, 180° slice-selection gradient; and Gro, readout gradient. The timing parameters, t_{kl}, are listed in table 1. The b-matrix is symmetric ($b_{ij} = b_{ji}$) so only six independent b-matrix elements are derived:

$b_{rr} = \gamma^2 \{ Grdp^2\, t_{22} + 2\, Gd_r\, Grdp\, t_{23} + 2\, Gc_r\, Grdp\, t_{24} + Gd_r^2\, t_{33} + Gc_r^2\, t_{44} + 2\, Gc_r\, Gd_r\, t_{34} + Gro^2\, t_{66} + 2\, Gro\, Grdp\, t_{26} \}$.

$b_{pp} = \gamma^2 \{Gd_p^2\, t_{33} + Gc_p^2\, t_{44} + 2\, Gc_p\, Gd_p\, t_{34}\}$

$b_{ss} = \gamma^2 \{Gsl^2\, (t_{11} + t_{15} + 1/4\, t_{55}) + 2\, Gsl\, Gsrf\, (t_{12} + 1/2\, t_{25}) + 2\, Gsl\, Gd_s\, (t_{13} + 1/2\, t_{35}) + 2\, Gsl\, Gc_s\, (t_{14} + 1/2\, t_{45}) + Gsrf^2\, t_{22} + 2\, Gd_s\, Gsrf\, t_{23} + 2\, Gc_s\, Gsrf\, t_{24} + Gd_s^2\, t_{33} + Gc_s^2\, t_{44} + 2\, Gc_s\, Gd_s\, t_{34}\}$.

$b_{rp} = \gamma^2 \{Grdp\, Gpe\, t_{22} + (Gd_p\, Grdp + Gd_r\, Gpe)\, t_{23} + (Gc_p\, Grdp + Gc_r\, Gpe)\, t_{24} + Gd_r\, Gd_p\, t_{33} + Gc_p\, Gc_r\, t_{44} + (Gc_p\, Gd_r + Gd_p\, Gc_r)\, t_{34} + Gro\, Gpe\, t_{26} \}$.

$b_{rs} = \gamma^2 \{Gsl\, Grdp\, (t_{12} + 1/2\, t_{25}) + Gsl\, Gd_r\, (t_{13} + 1/2\, t_{35}) + Gsl\, Gc_r\, (t_{14} + 1/2\, t_{45}) + Gro\, Gsl\, t_{16} + Gsrf\, Grdp\, t_{22} + (Gd_r\, Gsrf + Gd_s\, Grdp)\, t_{23} + (Gc_r\, Gsrf + Grdp\, Gc_s)\, t_{24} + Gro\, Gsrf\, t_{26} + Gd_s\, Gd_r\, t_{33} + Gc_r\, Gc_p\, t_{44} + (Gc_r\, Gd_s + Gd_r\, Gc_s)\, t_{34} \}$.

$b_{ps} = \gamma^2 \{Gsl\, Gpe\, (t_{12} + 1/2\, t_{25}) + Gsl\, Gd_p\, (t_{13} + 1/2\, t_{35}) + 2\, Gsl\, Gc_p\, (t_{14} + 1/2\, t_{45}) + Gsrf\, Gpe\, t_{22} + (Gd_s\, Gpe + Gd_p\, Gsrf)\, t_{23} + (Gc_p\, Gsrf + Gpe\, Gc_s)\, t_{24} + Gd_s\, Gd_p\, t_{33} + Gc_p\, Gc_s\, t_{44} + (Gc_p\, Gd_s + Gd_p\, Gc_s)\, t_{34} \}$.

Most b-factor calculations assume Gpe = 0, center line of Fourier space in the phase-encode direction (top of the echo).

Experimental Methods and Results:

We obtained 64x64 coronal images of an isotropic polyacrylamide gel phantom, with the diffusion gradient applied *in each of the three coordinate directions (read, phase, and* slice). Temperature was measured at 16.0°C in the phantom using Luxtron™ fiberoptic probes. A linear regression program was used to estimate the diffusion coefficient, standard deviation, and reduced chi-squared: $D_{read} = (1.473 \pm 0.048) \times 10^{-3}$ mm²/sec, $\chi^2 = 0.003$; $D_{phase} = (1.449 \pm 0.048) \times 10^{-3}$ mm²/sec, $\chi^2 = 0.0005$; and $D_{slice} = (1.427 \pm 0.041) \times 10^{-3}$ mm²/sec, $\chi^2 = 0.0008$. The spectroscopically obtained diffusion coefficient was $D = (1.43 \pm 0.06) \times 10^{-3}$ mm²/sec, $\chi^2 = 0.001$. These diffusion coefficients are within the experimental error. The goodness of fit and the similarity between diffusion coefficients estimated by imaging and spectroscopy show that the b-matrix components are calculated correctly. When we ignored the imaging gradients in calculating the b-factor, and estimated the diffusion coefficients we obtained: $D_{read} = 6.76 \times 10^{-3}$ mm²/sec, $\chi^2 = 6.57$; $D_{phase} = 6.34 \times 10^{-3}$ mm²/sec, $\chi^2 = 5.70$; and $D_{slice} = 6.12 \times 10^{-3}$ mm²/sec, $\chi^2 = 8.11$.

Discussion and Concluding Remarks:

Ignoring cross-terms in calculating the b-matrix introduces significant errors into the estimate of the apparent diffusion tensor. Additionally, in anisotropic media, the off-diagonal components of the b-matrix are required to estimate the components of the diffusion tensor, as described by Basser et al.[2], and should not be ignored whether or not there are specific diffusion gradients. These analytical expressions for the b-matrix permit the accurate determination of all the elements of the diffusion tensor. The linear fit of the imaging data demonstrate that the b-matrix components were calculated correctly in each of the three coordinate directions.

References:

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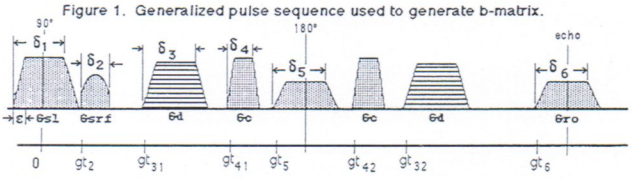
Figure 1. Generalized pulse sequence used to generate b-matrix.



t_{kl}	l = 1	2	3	4	5	6
		$\Delta_2 = TE - gt_{42}$	$\Delta_3 = gt_{32} - gt_{31}$			
1	$\frac{1}{4}[\delta_1^2(\tau_E - \frac{1}{3}\delta_1) + \frac{1}{30}\varepsilon^3 - \frac{1}{6}\delta_1\varepsilon^2]$,					
2	$\frac{1}{2}\delta_2\delta_1(\Delta_2 - \frac{1}{2}\delta_2)$,	$\frac{1}{3}\delta_2^2(\Delta_2 - \frac{1}{8}\delta_2)$,		$\Delta_4 = gt_{42} - gt_{41}$		
3	$\frac{1}{2}\delta_1\delta_3\Delta_3$,	$\frac{1}{2}\delta_2\delta_3\Delta_3$,	$[\delta_3^2(\Delta_3 - \frac{1}{3}\delta_3) + \frac{1}{30}\varepsilon^3 - \frac{1}{6}\delta_3\varepsilon^2]$,			
4	$\frac{1}{2}\delta_1\delta_4\Delta_4$,	$\frac{1}{2}\delta_2\delta_4\Delta_4$,	$\delta_3\delta_4\Delta_4$,	$[\delta_4^2(\Delta_4 - \frac{1}{3}\delta_4) + \frac{1}{30}\varepsilon^3 - \frac{1}{6}\delta_4\varepsilon^2]$,		
5	$\frac{1}{8}\delta_1(\delta_5^2 + \frac{1}{3}\varepsilon^2)$,	$\frac{1}{4}\delta_2(\delta_5^2 + \frac{1}{3}\varepsilon^2)$,	$\frac{1}{4}\delta_3(\delta_5^2 + \frac{1}{3}\varepsilon^2)$,	$\frac{1}{4}\delta_4(\delta_5^2 + \frac{1}{3}\varepsilon^2)$,	$\frac{1}{2}[\frac{1}{6}\delta_5^3 + \frac{1}{30}\varepsilon^3]$,	
6	$-\frac{1}{16}\delta_1(\delta_6^2 + \frac{1}{3}\varepsilon^2)$,	$-\frac{1}{8}\delta_2(\delta_6^2 + \frac{1}{3}\varepsilon^2)$,	0,	0,	0,	$\frac{1}{4}[\frac{1}{6}\delta_6^3 + \frac{1}{30}\varepsilon^3]$,

Table 1, Time parameters used in calculating the b-matrix, gt_i is time gradient pulse turns on during pulse sequence.